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Behavior of chemotaxis model with density-dependent sensitivity

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Abstract. In this paper we study a Keller-Segel-type chemotaxis model with reproduction term under no-flux boundary conditions in R^n, where the kinetics term of the second equation is nonlinear.Assuming some growth conditions for the kinetics term function, some priori estimates are established for the solution of the problem. Furthermore, it is shown that there is a bounded global solution to the problem by the priori estimate.

Key words. Chemotaxis model, priori estimates, local existence.

1. Introduction

Chemotaxis phenomenon is quite common phenomenon in bio-system. The first chemotaxis equation was introduced by Keller and Segal [1] to describe the aggregation of slime mold amoebaedue to an attractive chemical substance.

In this paper the following chemotaxis model is discussed :

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \nabla \cdot \left(\frac{u}{1+\varepsilon u} \nabla w \right), & x \in \Omega, \quad t > 0 \\ \frac{\partial w}{\partial t} = \Delta w + g(u) - w, & x \in \Omega, \quad t > 0 \\ \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \quad t > 0 \\ u(x,0) = u_0(x), w(x,0) = w_0(x). & x \in \Omega. \end{cases} \tag{1}$$

where$u$represents the density or population of a biological species, which could be a cell, a germ, or an insect, while$w$represents an attractive resource of the species. $g(u)$is the production function of$w$, it is always assumed in the paper that$g(u) = u^{\gamma_0}$, $1 \le \gamma_0 < n + 1/n$. $\varepsilon$is positive constant, Ω is a subset of R^n.

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2. Preliminary

In this paper some well-known inequalities and embedding results that will be used in the sequel are presented.

Lemma 2.1[2]. If $p, q \geq 1$ and$p(n-q) < nq$, then, for any $r \in (0,p)$, there exists $a = \frac{\frac{n}{r}-\frac{n}{p}}{1-\frac{n}{q}+\frac{n}{r}}$, such that $\|u\|_{L^p_{(\Omega)}} \leq c\|u\|^a_{W^{1,q}} \|u\|^{1-a}_{L^r_{(\Omega)}}$

Lemma 2.2[2]. If $1 \leq q \leq p < +\infty$ and$f \in L^q(\Omega)$, then,

$$\left\|e^{t\Delta} f\right\|_p \leq (4\pi t)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_q, \left\|\nabla e^{t\Delta} f\right\|_p \leq ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}} \|f\|_q$$

Lemma 2.3[3]. Let $A_p = -\Delta$ and$D(A_p) = \left\{\varphi \in W^{2,p}(\Omega) \middle| \frac{\partial\varphi}{\partial n}|_{\partial\Omega} = 0\right\}$, if$1 < p < +\infty$ then,

$$\left\|(A+1)^\beta e^{-t(A+1)} u\right\|_{L^p(\Omega)} \leq ct^{-\beta}\|u\|_{L^p(\Omega)}$$

Lemma 2.4[4]. Let $\beta > 0$ and$1 < p < +\infty$, for any $\varepsilon > 0$,there exists$c(\varepsilon) > 0$ such that

$$\left\|(-\Delta+1)^\beta e^{-t\Delta} \nabla \cdot w\right\|_{L^p(\Omega)} \leq c(\varepsilon) t^{-\beta-\frac{1}{2}-\varepsilon} \|w\|_{L^p(\Omega)}$$

3. Local existence and uniqueness

The local existence of a solution to system (1) is discussed in this section.

Theorem 3.1. Suppose$0 \leq u_0(x) \in C^0(\Omega)$, $0 \leq w_0(x) \in W^{1,p}(\Omega)$, $p > n$,then there is a $T > 0$ (depending on $\|u_0\|_{C^0(\Omega)}$, $\|u_0\|_{W^{1,p}(\Omega)}$)such that there is a unique nonnegative solution$(u(x,t), w(x,t))$ to (1) in $[0,T)$ and satisfying

$$u(x,t) \in C\left([0,T); C^0(\Omega)\right) \bigcap C^{2,1}(\Omega; [0,T)), w(x,t) \in C\left([0,T); W^{1,p}(\Omega)\right) \bigcap C^{2,1}(\Omega; [0,T))$$

Proof: Choose $T \in (0,1)$ and $R > 0$ to be fixed. In the Banachspace $X = C^0\left([0,T); C^0(\Omega)\right) \times C^0\left([0,T); W^{1,p}(\Omega)\right)$, we consider the bounded closed set

$$S := \{(u,w) \in X \,|\|(u,w)\|_X \leq R\}$$

Let

$$\psi(u,w)(t) = \begin{pmatrix} \psi_1(u,w) \\ \psi_2(u,w) \end{pmatrix} = \begin{pmatrix} e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}[\nabla\left(\frac{u}{1+\varepsilon u}\nabla w\right)]ds \\ e^{t(\Delta-1)}w_0 + \int_0^t e^{(t-s)(\Delta-1)}u^{\gamma_0}(s)\,ds \end{pmatrix}$$

Next we prove $\psi$is a contraction from$S$into itself, when$T$small enough and R sufficiently large. We denoted the operator$-\Delta$by$A$. Then let $n/2q < \beta < 1/2$and$0 < \varepsilon_0 < 1/2 - \beta$, by the Lemma 2.2-2.3, for any $t \in [0,T)$,

$$\begin{aligned}\|\psi_1(u,w)\|_{C^0} &\le \left\|e^{-tA}u_0\right\|_{C^0} + c\int_0^t \left\|(A+1)^\beta e^{-(t-s)A}\nabla\left(\tfrac{u}{1+\varepsilon u}\nabla w\right)\right\|_{L^q} ds \\ &\le \|u_0(x)\|_{C^0} + c\int_0^t (t-s)^{-\frac{1}{2}-\beta-\varepsilon_0}\left\|\tfrac{u}{1+\varepsilon u}\nabla w\right\|_{L^q} ds\end{aligned}$$

Since $\left\|\frac{u}{1+\varepsilon u}\nabla w\right\|_{L^q} \le \frac{1}{\varepsilon}\|\nabla w\|_{L^q} \le \frac{1}{\varepsilon}R$, then we have that
$\int_0^t (t-s)^{-\frac{1}{2}-\beta-\varepsilon_0}\left\|\frac{u}{1+\varepsilon u}\nabla w\right\|_{L^q} ds \le RT^{\frac{1}{2}-\beta-\varepsilon_0}$.

$$\|\psi_1(u,w)\|_{C^0} \le \|u_0\|_{C^0} + cRT^{\frac{1}{2}-\beta-\varepsilon_0} \tag{2}$$

By Lemma 2.3, for any $t\in[0,T)$,

$$\begin{aligned}\|\psi_2(u,w)\|_{W^{1,p}} &\le \left\|e^{-t(A+1)}w_0\right\|_{W^{1,p}} + \int_0^t \left\|e^{-(t-s)(A+1)}u^{\gamma_0}(s)\right\|_{W^{1,p}} ds \\ &\le \|w_0\|_{W^{1,p}} + c\int_0^t (t-s)^{-\gamma}\|u(s)\|_{L^p} ds \\ &\le \|w_0\|_{W^{1,p}} + cRT^{1-\gamma}\end{aligned} \tag{3}$$

here $\gamma\in(\frac{1}{2},1)$. That (2) and (3) implies that$\psi S\subset S$for any fixed positive R and T small enough. Now we show ψ is a contract operator from $S$to S. For any$(u,w),(\overline{u},\overline{w})\in S,\ t\in[0,T)$

$$\begin{aligned}\|\psi_1(u,w)-\psi_1(\overline{u},\overline{w})\|_{C^0} &\le c\int_0^t \left\|(A+1)^\beta e^{-(t-s)A}[\nabla\left(\tfrac{u}{1+\varepsilon u}\nabla w\right)-\nabla\left(\tfrac{\overline{u}}{1+\varepsilon\overline{u}}\nabla\overline{w}\right)]\right\|_{L^q} ds \\ &\le c\int_0^t (t-s)^{-\frac{1}{2}-\beta-\varepsilon_0}\left\|\tfrac{u}{1+\varepsilon u}\nabla w-\tfrac{\overline{u}}{1+\varepsilon\overline{u}}\nabla\overline{w}\right\|_{L^q} ds \\ &\le cR^2\left(T^{-\frac{1}{2}-\beta-\varepsilon_0}+Ce^{(1-\mu)T}T^{\frac{1}{2}-\beta}\right)\|(u,w)-(\overline{u},\overline{w})\|_X\end{aligned} \tag{4}$$

$$\begin{aligned}\|\psi_2(u,w)-\psi_2(\overline{u},\overline{w})\|_{W^{1,p}} &\le c\int_0^t \left\|(A+1)^\gamma e^{-(t-s)(A+1)}[u^{\gamma_0}(s)-\overline{u}^{\gamma_0}(s)]\right\|_{L^p} ds \\ &\le c\gamma_0 R^{\gamma_0-1}\int_0^t (t-s)^{-\gamma}\|u(s)-\overline{u}(s)\|_{L^p} ds \\ &\le c\gamma_0 R^{\gamma_0-1}T^{1-\gamma}\|(u,w)-(\overline{u},\overline{w})\|_X\end{aligned} \tag{5}$$

That (4) and (5) implies that ψ is a contractive mapping if$T$ is sufficiently small. By Banach's fixed point theorem , we obtain there exists a unique fixed point $(u,w)\in X$, which is just a local solution to (1). Since $u_0(x)\ge 0; w_0(x)\ge 0$,by the Comparison principle, we known that for any $t\in[0,T)$,$u(x,t)\ge 0; w(x,t)\ge 0$,

From above analysis and $w_0(x)\in W^{1,p}(\Omega)\to C^0(\Omega)$,then$w_0(x)\in W^{1,p}(\Omega), t\in[0,T)$. Relying on this and $u(x,t)\in C\left([0,T);C^0(\Omega)\right)$, by the regularity argument and Schauder's estimates. The proof of the theoremis completed.

$$u(x,t)\in C\left([0,T);C^0(\Omega)\right)\in C\left([0,T);W^{1,p}(\Omega)\right)\bigcap C^{2,1}(\Omega;[0,T))\bigcap C\left((0,T);C^3(\Omega)\right)$$

4. Priori estimates of the system

Integrating the fist equation of (1) onΩ, and by the boundary conditions,which imply that

Lemma 4.1 Suppose that$0 \leq u_0(x) \in L^2(\Omega)$,$(u,\ w)$isthe solution of (1), then for any $t \in [0, T)$, $\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx$

By the Gronwall's lemma and lemma 2.1, the following theorem can be proved.

Theorem 4.1. Suppose that$\gamma_0 \in [1, n+1/n), 0 \leq u_0(x) \in C^0(\Omega)$, $0 \leq w_0(x) \in W^{1,p}(\Omega)$, $(u,\ w)$isa local solution of (1) in $[0, T)$. Then there exists a constant $\nu > 0$, such that

$$\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 \leq c\left(1 + e^{-\upsilon t}\left(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2\right)\right),\ t \in [0, T)$$

5. Conclusion

In this paper, we have proved the local existence of the solution to system (1) on the condition that $\gamma_0 \in [1, n+1/n)$, and some priori estimates of the system is given. From the proses of the proof, it is easy to know that the similar results can be obtained for $u \leq g(u) \leq u^{\gamma_0}$.

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